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# Divergence terms in the supertrace heat asymptotics for the de Rham complex on a manifold with boundary

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## Abstract

We use invariance theory to determine the coefficient  $a_{m+1,m}^{d+\delta}$  in the supertrace for the twisted de Rham complex with absolute boundary conditions.

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## 1. Introduction

Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m$  with smooth, non-empty boundary  $\partial M$ . Let  $\phi \in C^\infty(M)$  be an auxiliary smooth function called the dilaton. Let  $d_\phi := e^{-\phi} d e^\phi$  and let  $\delta_{\phi,g} := e^\phi \delta_g e^{-\phi}$  be the twisted exterior derivative and the coderivative, respectively, on the space of smooth differential forms. The *twisted* or *Witten* Laplacian is given by

$$\Delta_{\phi,g}^p := d_\phi \delta_{\phi,g} + \delta_{\phi,g} d_\phi \quad \text{on } C^\infty(\Lambda^p(M)).$$

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This operator appears in the study of quantum  $p$  form fields interacting with a background dilaton [13,21]. It has also been used in supersymmetric quantum mechanics [1] and in Morse theory [23].

We impose *absolute boundary conditions*  $\mathcal{B}_a$ , see [12] for details. Let  $\Delta_{\phi,g,\mathcal{B}_a}^p$  be the associated realization. We need not consider relative boundary conditions  $\mathcal{B}_r$  as the Hodge  $\star$  operator intertwines  $\Delta_{\phi,g,\mathcal{B}_a}^p$  and  $\Delta_{-\phi,g,\mathcal{B}_r}^{m-p}$  if  $M$  is orientable [13]. These boundary conditions are motivated by the Hodge–de Rham theorem which shows

$$\ker(\Delta_{\phi,g,\mathcal{B}_a}^p) = H^p(M).$$

The fundamental solution  $e^{-t\Delta_{\phi,g,\mathcal{B}_a}^p}$  of the heat equation is an infinitely smoothing operator which is of trace class. Let  $f \in C^\infty(M)$  be a smooth smearing function. Work of Greiner [14] and Seeley [19] shows there is a complete asymptotic expansion:

$$\text{Tr}_{L^2}(f e^{-t\Delta_{\phi,g,\mathcal{B}_a}^p}) \sim \sum_{n \geq 0} a_{n,m}(f, \Delta_{\phi,g}^p, \mathcal{B}_a) t^{(n-m)/2} \quad \text{as } t \downarrow 0.$$

The *heat trace invariants*  $a_{n,m}(\cdot)$  are locally computable. Let  $\nabla_{e_m}^k f$  be the  $k$ th covariant derivative of  $f$  with respect to the inward unit normal  $e_m$  on  $\partial M$ . Let  $dx$  and  $dy$  be the Riemannian measures on  $M$  and on  $\partial M$ , respectively. There exist local invariants  $a_{n,m}(x, \Delta_{\phi,g}^p)$  and  $a_{n,m,k}(y, \Delta_{\phi,g}^p, \mathcal{B}_a)$  so that

$$\begin{aligned} a_{n,m}(f, \Delta_{\phi,g}^p, \mathcal{B}_a) &= \int_M f(x) a_{n,m}(x, \Delta_{\phi,g}^p) dx \\ &+ \sum_k \int_{\partial M} \nabla_{e_m}^k f(y) \cdot a_{n,m,k}(y, \Delta_{\phi,g}^p, \mathcal{B}_a) dy. \end{aligned}$$

The interior invariants vanish if  $n$  is odd; the boundary invariants are generically non-zero for all  $n \geq 1$ . The presence of the smearing function  $f$  localizes the problem and permits the recovery of divergence terms which would otherwise be lost. The presence of terms involving  $\nabla_{e_m}^k f$  shows the kernel function for the fundamental solution of the heat equation behaves asymptotically like a distribution near the boundary as  $t \downarrow 0$ . Define the local *supertrace heat asymptotics* by setting:

$$\begin{aligned} a_{n,m}^{d+\delta}(\phi, g)(x) &:= \sum_p (-1)^p a_{n,m}(x, \Delta_{\phi,g}^p), \\ a_{n,m,k}^{d+\delta}(\phi, g)(y) &:= \sum_p (-1)^p a_{n,m,k}(y, \Delta_{\phi,g}^p, \mathcal{B}_a). \end{aligned}$$

Let  $\chi(M)$  be the Euler–Poincaré characteristic of  $M$ . If  $f = 1$  and if  $\phi$  satisfies Neumann boundary conditions, then [13]:

$$\sum_p (-1)^p \text{Tr}_{L^2}(e^{-t\Delta_{\phi,g,\mathcal{B}_a}^p}) = \chi(M).$$

Equating terms in the asymptotic series yields:

$$\int_M a_{n,m}^{d+\delta}(\phi, g)(x) dx + \int_{\partial M} a_{n,m,0}^{d+\delta}(\phi, g)(y) dy = \begin{cases} \chi(M) & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{1.1}$$

The local index density has been computed in this setting [13]. Let indices  $i, j, \dots$  range from 1 to  $m$  and index a local orthonormal frame for the tangent bundle of  $M$ . Let  $R_{ijkl}$  be the associated components of the Riemann curvature tensor with the sign convention that  $R_{1221} = +1$  on the unit sphere  $S^2 \subset \mathbb{R}^3$ . Near the boundary, normalize the choice of the orthonormal frame so  $e_m$  is the inward unit geodesic normal. Let indices  $a, b, \dots$  range from 1 to  $m - 1$  and index the induced orthonormal frame for the tangent bundle of the boundary. Let  $L_{ab}$  be the components of the second fundamental form.

We adopt the Einstein convention and sum over repeated indices. Let

$$\varepsilon_U^V := g(e_{u_1} \wedge \dots \wedge e_{u_\mu}, e_{v_1} \wedge \dots \wedge e_{v_\mu})$$

be the totally anti-symmetric tensor. Let  $I$  and  $J$  be  $m$  tuples of indices indexing an orthonormal frame for  $T(M)$  and let  $A$  and  $B$  be  $m - 1$  tuples of indices indexing an orthonormal frame for  $T(\partial M)$ . Set

$$\begin{aligned} \mathcal{R}_{J,S}^{I,t} &:= R_{i_s i_{s+1} j_{s+1} j_s} \dots R_{i_{t-1} i_t j_t j_{t-1}}, & \mathcal{R}_{B,S}^{A,t} &:= R_{a_s a_{s+1} b_{s+1} b_s} \dots R_{a_{t-1} a_t b_t b_{t-1}}, \\ \mathcal{L}_{B,S}^{A,t} &:= L_{a_s b_s} \dots L_{a_t b_t}. \end{aligned}$$

Since the empty product is 1, we set  $\mathcal{R}_{J,S}^{I,t} = 1$ ,  $\mathcal{R}_{B,S}^{A,t} = 1$ , and  $\mathcal{L}_{B,S}^{A,t} = 1$  if  $t < s$ .

We refer to [13] for the proof of the following result. It establishes vanishing theorems which generalize previous results of [2,10,11,18] to the twisted setting. It also identifies the local index density in the twisted setting.

**Theorem 1.1.**

- (1) If  $n$  is odd or if  $n < m$ , then  $a_{n,m}^{d+\delta}(\phi, g) = 0$ .
- (2) If  $m$  is odd, then  $a_{n,m}^{d+\delta}(0, g) = 0$  for any  $n$ .
- (3) If  $n - k < m$ , then  $a_{n,m,k}^{d+\delta}(\phi, g) = 0$ .
- (4)  $a_{2\bar{m},2\bar{m}}^{d+\delta}(\phi, g) = 1/(\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!) \varepsilon_J^I \mathcal{R}_{J,1}^{I,m}$ .
- (5)  $a_{m,m,0}^{d+\delta}(\phi, g) = \sum_k 1/(\pi^k 8^k k!(m - 1 - 2k)! \text{vol}(S^{m-1-2k})) \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,m-1}$ .

The fact that the local index density is not dependent on the dilaton field has important physical consequences [13]. One can also combine Eq. (1.1) with Theorem 1.1 to obtain a heat equation proof of the Chern–Gauss–Bonnet theorem [8,9] for manifolds

with boundary:

$$\begin{aligned} \chi(M^{2\bar{m}}) &= \int_M \frac{1}{\pi^{\bar{m}} 8^{\bar{m}} \bar{m}!} \varepsilon^J \mathcal{R}_{J,1}^{l,m} dx \\ &+ \sum_k \int_{\partial M} \frac{1}{\pi^k 8^k k! (2\bar{m} - 1 - 2k)! \text{vol}(S^{2\bar{m}-1-2k})} \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,2\bar{m}-1} dy, \\ \chi(M^{2\bar{m}+1}) &= \sum_k \int_{\partial M} \frac{1}{\pi^k 8^k k! (2\bar{m} - 2k)! \text{vol}(S^{2\bar{m}-2k})} \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,2\bar{m}} dy. \end{aligned}$$

By Theorem 1.1, the first non-trivial ‘divergence’ terms can first arise in the supertrace when  $n = m + 1$ . Let ‘;’ and ‘:’ denote multiple covariant differentiation with respect to the Levi–Civita connections on  $M$  and on  $\partial M$ , respectively. By Theorem 1.1,  $a_{m+1,m}^{d+\delta}(\phi, g) = 0$  if  $m$  is even. Furthermore  $a_{m+1,m,k}^{d+\delta}(\phi, g) = 0$  if  $k \geq 2$ . The following is the main result of this paper:

**Theorem 1.2.**

- (1)  $a_{2\bar{m}+2,2\bar{m}+1}^{d+\delta}(\phi, g) = (1/\sqrt{\pi}\pi^{\bar{m}}8^{\bar{m}}\bar{m}!) \varepsilon^J \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{l,m}$ .
- (2)  $a_{m+1,m,0}^{d+\delta}(\phi, g) = \sum_k (1/\sqrt{\pi}\pi^k 8^k k! \text{vol}(S^{m-2k-2})(m - 2k - 2)!) \varepsilon_B^A \phi_{;a_1 b_1} \mathcal{R}_{B,2}^{A,2k+1} \mathcal{L}_{B,2k+2}^{A,m-1} + \sum_{2k < m-3} (1/2\sqrt{\pi}\pi^k 8^k k! \text{vol}(S^{m-2k-2})(m - 2k - 2)!) \varepsilon_B^A \{ \mathcal{R}_{B,1}^{A,2k} R_{a_{2k+1} a_{2k+2} b_{2k+2} m} \mathcal{L}_{B,2k+3}^{A,m-1} \}_{;b_{2k+1}}$ .
- (3)  $a_{m+1,m,1}^{d+\delta}(\phi, g) = \sum_k (\sqrt{\pi}/8^k \pi^k k! \text{vol}(S^{m-2k})(m - 2k)!) \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,m-1}$ .

Let  $M$  be a closed manifold. The local index density for the untwisted de Rham complex was identified in dimension 2 by McKean and Singer [16] and in arbitrary dimensions by Atiyah et al. [2], by Gilkey [10], and by Patodi [18]. The case of manifolds with boundary was studied in [11]. We also refer to [3,4,17] for other treatments of the local index theorem.

Patodi’s approach involved a direct calculation analyzing cancellation formulas for the fundamental solution of the heat equation. Atiyah et al. used invariance theory to identify the local index density for the twisted signature and twisted spin complexes. They then expressed the de Rham complex locally in terms of the spin complex twisted by a suitable coefficient bundle. Neither of these approaches seems particularly well adapted to the twisted setting. In particular, since the operator  $d_\phi$  relies on the  $\mathbb{Z}$  grading of the de Rham complex, it is not described in terms of an operator on the twisted signature or spin complexes. Thus we choose in [13] to generalize the approach of [10] to determine the local index density for the twisted de Rham complex.

There are explicit combinatorial formulas [6,7,15] for the heat trace invariants of order  $n \leq 5$ , see the discussion in Section 2 for further details. However, these formulas become very complicated and it seems hopeless to prove Theorem 1.2 by an explicit computation.

The approach taken by Gilkey in [10] suffered from the disadvantage that the techniques involved were rather ad hoc and cumbersome as they did not make full use of the machinery of invariance theory developed by Weyl [22]. In the present paper, we use both the first and second main theorems of invariance theory; this is the crucial new feature of our

analysis. Let

$$\begin{aligned} \mathcal{E}_{m+1,m} &:= \varepsilon_I^J \phi_{;i_1 j_1} \mathcal{R}_{J,2}^{I,m}, & \mathcal{F}_{m-1,m}^k &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \mathcal{L}_{B,2k+1}^{A,m-1}, \\ \mathcal{F}_{m,m}^{1,k} &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \phi_{;a_{2k+1} b_{2k+1}} \mathcal{L}_{B,2k+2}^{A,m-1}, & \mathcal{F}_{m,m}^{2,k} &:= \varepsilon_B^A \mathcal{R}_{B,1}^{A,2k} \phi_{;a_{2k+1}} \phi_{;b_{2k+1}} \mathcal{L}_{B,2k+2}^{A,m-1}, \\ \mathcal{F}_{m,m}^{3,k} &:= \varepsilon_B^A \{ \mathcal{R}_{B,1}^{A,2k} R_{a_{2k+1} a_{2k+2} b_{2k+2} m} \mathcal{L}_{B,2k+3}^{A,m-1} \}_{;b_{2k+1}}. \end{aligned}$$

**Lemma 1.3.** *There exist universal constants so that:*

- (1) *If  $m$  is odd, then  $a_{m+1,m}^{d+\delta}(\phi, g) = c_{m+1,m} \mathcal{E}_{m+1,m}$ .*
- (2)  $a_{m+1,m,1}^{d+\delta}(\phi, g) = \sum_k c_{m+1,m,1}^k \mathcal{F}_{m-1,m}^k$ .
- (3)  $a_{m+1,m,0}^{d+\delta}(\phi, g) = \sum_{i,k} c_{m+1,m,0}^{i,k} \mathcal{F}_{m,m}^{i,k}$ .

This reduces the proof of Theorem 1.2 to the evaluation of the unknown universal coefficients. Here is a brief guide to the remainder of the paper. In Section 2, we review the properties of the heat trace invariants which we will need. In Section 3, we use invariance theory to establish Lemma 1.3. In Section 4, we employ product formulas, special case calculations, and functorial properties to derive some technical results concerning the universal coefficients of Lemma 1.3. We then combine these results to complete the proof of Theorem 1.2 in Section 5.

## 2. Formulas for the heat trace asymptotics

Let  $D$  be an arbitrary operator of Laplace type on a vector bundle  $V$ . There is a canonical connection [12]  $\nabla$  on  $V$  which we use to differentiate tensors of all types and a canonical endomorphism  $E$  of  $V$  so that

$$Du = -(u_{;ij} + Eu).$$

We impose mixed boundary conditions. Let  $\chi$  be an endomorphism of  $V|_{\partial M}$  so  $\chi^2 = 1$ . Decompose  $\chi = \Pi_+ - \Pi_-$  where  $\Pi_{\pm} := (1/2)(\text{Id} \pm \chi)$  are the projections on the  $\pm 1$  eigenspaces of  $\chi$ . Let  $S$  be an auxiliary endomorphism of  $\Pi_+$ . We extend  $\chi$  and  $S$  to be parallel with respect to the geodesic normal vector field  $e_m$  near  $\partial M$ . We impose Robin boundary conditions on  $V_+ := \text{Range}(\Pi_+)$  and Dirichlet boundary conditions on  $V_- := \text{Range}(\Pi_-)$  to define the mixed boundary operator:

$$\mathcal{B} := \{ \Pi_+(\nabla_{e_m} + S) \oplus \Pi_- \}_{|\partial M}.$$

Let  $\Omega_{ij}$  be the components of the curvature endomorphism defined by  $\nabla$ . We refer to [6] for the proof of the following result which expresses the heat trace asymptotics in terms of this formalism for  $n \leq 3$ :

**Lemma 2.1.**

- (1)  $a_0(f, D, \mathcal{B}) = (4\pi)^{-m/2} \int_M \text{Tr}(f \text{Id}) \, dx$ .
- (2)  $a_1(f, D, \mathcal{B}) = (4\pi)^{-(m-1)/2} (1/4) \int_{\partial M} \text{Tr}(f\chi) \, dy$ .

- (3)  $a_2(f, D, \mathcal{B}) = (4\pi)^{-m/2}(1/6) \int_M \text{Tr}\{f(6E + R_{ijji} \text{Id})\} dx + (4\pi)^{-m/2}(1/6) \int_{\partial M} \text{Tr}\{f(2L_{aa} \text{Id} + 12S) + 3f_{;m} \chi\} dy.$
- (4)  $a_3(f, D, \mathcal{B}) = (4\pi)^{-(m-1)/2}(1/384) \int_{\partial M} \text{Tr}\{f(96\chi E + 16\chi R_{ijji} + 8\chi R_{amam} + [13\Pi_+ - 7\Pi_-]L_{aa}L_{bb} + [2\Pi_+ + 10\Pi_-]L_{ab}L_{ab} + 96SL_{aa} + 192S^2 - 12\chi_{;a}\chi_{;a}) + f_{;m}([6\Pi_+ + 30\Pi_-]L_{aa} + 96S) + 24\chi f_{;mm}\} dy.$

Similar formulas are available [6,7,15,20] for  $n = 4, 5$ . What is crucial to our analysis, however, is the general form of these expressions. They are the trace of certain non-commutative polynomials in the covariant derivatives of the variables  $\{R, E, \Omega, S, L, \chi\}$  with indices contracted in pairs.

To apply Lemma 2.1 to the setting at hand, we must identify the structures which are involved for the twisted Laplacian. Let  $\epsilon_i : \omega \rightarrow e_i \wedge \omega$  be left exterior multiplication by the covector  $e_i$  and let  $i_i$  be the dual operator, left interior multiplication by  $e_i$ . Let  $\gamma_i = \epsilon_i - i_i$  give the associated Clifford module structure on the exterior algebra. Extend the Levi–Civita connection to act on tensors of all types and let  $\Omega_{ij}$  be the associated curvature operator.

**Lemma 2.2.**

- (1)  $\Delta_{\phi,g} = \Delta_g + \phi_{;i}\phi_{;i} \cdot \text{Id} + \phi_{;ji}(\epsilon_i i_j - i_j \epsilon_i).$
- (2) *The Levi–Civita connection is the connection associated to  $\Delta_{\phi,g}$ .*
- (3)  $E_{\phi,g} := -(1/2)\gamma_i \gamma_j \Omega_{ij} - \phi_{;i}\phi_{;i} - \phi_{;ji}(\epsilon_i i_j - i_j \epsilon_i)$  is the endomorphism for  $\Delta_{\phi,g}$ .
- (4) *Absolute boundary conditions are defined by taking*

$$\chi := \left\{ \begin{array}{ll} +1 & \text{on } \Lambda(\partial M) \\ -1 & \text{on } \Lambda(\partial M)^\perp \end{array} \right\} \quad \text{and} \quad S := \left\{ \begin{array}{ll} -L_{ab}\epsilon_b i_a & \text{on } \Lambda(\partial M) \\ 0 & \text{on } \Lambda(\partial M)^\perp \end{array} \right\}.$$

- (5)  $\chi_{;a} = 2L_{ab}(\epsilon_b i_m + \epsilon_m i_b).$

**Proof.** The classical formula  $d + \delta_g = \epsilon_i \nabla_{e_i} - i_j \nabla_{e_j}$  extends to the twisted setting:

$$d\phi + \delta_{\phi,g} = \epsilon_i \nabla_{e_i} + \epsilon_i \phi_{;i} - i_i \nabla_{e_i} + i_i \phi_{;i}.$$

We use the commutation rules  $\epsilon_i i_j + i_j \epsilon_i = \delta_{ij}$ , the fact that  $\nabla \epsilon = 0$ , and the fact that  $\nabla i = 0$  to prove Assertion (1) by computing:

$$\begin{aligned} \Delta_{\phi,g} &= \Delta_g + \epsilon_i \nabla_{e_i} i_j \phi_{;j} + i_j \phi_{;j} \epsilon_i \nabla_{e_i} - i_i \nabla_{e_i} \epsilon_j \phi_{;j} - \epsilon_j \phi_{;j} i_i \nabla_{e_i} + (\epsilon_i i_j + i_j \epsilon_i) \phi_{;i} \phi_{;j} \\ &= \Delta_g + (\epsilon_i i_j + i_j \epsilon_i - i_i \epsilon_j - \epsilon_j i_i) \phi_{;j} \nabla_{e_i} + (\epsilon_i i_j - i_i \epsilon_j) \phi_{;ji} + \phi_{;i} \phi_{;i} \\ &= \Delta_g + (\epsilon_i i_j - i_i \epsilon_j) \phi_{;ji} + \phi_{;i} \phi_{;i}. \end{aligned}$$

This shows that the associated connection does not depend on  $\phi$  and hence is the Levi–Civita connection [12]. Since the standard Weitzenböck formulas yield  $E(\Delta_g) = -(1/2)\gamma_i \gamma_j \Omega_{ij}$ , Assertion (3) follows.

We refer to [6] for the proof of Assertion (4). Let  $\omega_+ := e^{a_1} \wedge \dots \wedge e^{a_\ell}$  and let  $\omega_- := e^m \wedge \omega_+$ . We then have  $\chi \omega_\pm = \pm \omega_\pm$ . We use Assertion (4) to prove Assertion (5)

by computing:

$$\begin{aligned}
 (\nabla_{e_a} \chi - \chi \nabla_{e_a}) \omega_+ &= (\Gamma_{abc} \epsilon_c \mathfrak{i}_b + \Gamma_{abm} \epsilon_m \mathfrak{i}_b - \Gamma_{abc} \epsilon_c \mathfrak{i}_b + \Gamma_{abm} \epsilon_m \mathfrak{i}_b) \omega_+ = 2L_{ab} \epsilon_m \mathfrak{i}_b \omega_+, \\
 (\nabla_{e_a} \chi - \chi \nabla_{e_a}) \omega_- &= (-\Gamma_{abc} \epsilon_c \mathfrak{i}_b - \Gamma_{amb} \epsilon_b \mathfrak{i}_m + \Gamma_{abc} \epsilon_c \mathfrak{i}_b - \Gamma_{amb} \epsilon_b \mathfrak{i}_m) \omega_- = 2L_{ab} \epsilon_b \mathfrak{i}_m \omega_-.
 \end{aligned}$$

□

We now discuss functorial properties of the supertrace asymptotics.

**Lemma 2.3.**

- (1) On the circle,  $a_{2,1}^{d+\delta} = (1/\sqrt{\pi})\phi;_{11}$ .
- (2) We have  $a_{n,m}^{d+\delta}(\phi, g)(x) = (-1)^m a_{n,m}^{d+\delta}(-\phi, g)(x)$ .
- (3) We have  $\int_{\partial M} a_{m+1,m,0}^{d+\delta}(0, g) dy = 0$ .
- (4) Let  $(M, \phi, g) := (M_1 \times M_2, \phi_1 + \phi_2, g_1 + g_2)$  where  $\partial M_1 = \emptyset$ . Then
  - (a)  $a_{n,m}^{d+\delta}(\phi, g) = \sum_{n_1+n_2=n} a_{n_1,m_1}^{d+\delta}(\phi_1, g_1) \cdot a_{n_2,m_2}^{d+\delta}(\phi_2, g_2)$ ,
  - (b)  $a_{n,m,k}^{d+\delta}(\phi, g) = \sum_{n_1+n_2=n} a_{n_1,m_1}^{d+\delta}(\phi_1, g_1) \cdot a_{n_2,m_2,k}^{d+\delta}(\phi_2, g_2)$ .

**Proof.** Assertion (1) follows from Lemma 2.1 (3) and from Lemma 2.2 (3).

Since the interior invariants  $a_{n,m}^{d+\delta}(\phi, g)$  are local, we may suppose without loss of generality that  $M$  is a closed orientable manifold in the proof of Assertion (2). Let  $\tilde{\star}_g$  be the normalized Hodge operator defined by the metric. Then, the normalizations having taken into account the sign conventions, the usual intertwining relations extend to the twisted context to show

$$\tilde{\star}_g^2 = \text{id}, \quad \tilde{\star}_g d_\phi \tilde{\star}_g = \delta_{-\phi,g}, \quad \tilde{\star}_g \delta_{\phi,g} \tilde{\star}_g = d_{-\phi}.$$

Assertion (2) now follows from the intertwining relationship:

$$\tilde{\star}_g \Delta_{\phi,g}^p \tilde{\star}_g = \Delta_{-\phi,g}^{m-p}.$$

We note that  $\tilde{\star}_g$  intertwines absolute and relative boundary conditions; thus we can not conclude a similar equivariance property for the boundary invariants.

We use Theorem 1.1 to see that  $a_{m+1,m}^{d+\delta}(0, g) = 0$  regardless of the parity of  $m$ . As the interior invariant vanishes pointwise, the boundary integral vanishes by Eq. (1.1).

To prove Assertion (4), we decompose

$$\Lambda(M) = \Lambda(M_1) \otimes \Lambda(M_2), \quad d_\phi = d_1 + d_2, \quad \delta_{\phi,g} = \delta_1 + \delta_2,$$

where, on  $C^\infty(\Lambda^p(M_1) \otimes \Lambda^q(M_2))$ , we have

$$\begin{aligned}
 d_1 &:= d_{\phi_1} \otimes \text{Id}, & d_2 &:= (-1)^p \text{Id} \otimes d_{\phi_2}, \\
 \delta_1 &:= \delta_{\phi_1,g_1} \otimes \text{Id}, & \delta_2 &:= (-1)^p \text{Id} \otimes \delta_{\phi_2,g_2}.
 \end{aligned}$$

Consequently these operators satisfy the commutation relations:

$$d_1 d_2 + d_2 d_1 = 0, \quad d_1 \delta_2 + \delta_2 d_1 = 0, \quad \delta_1 d_2 + d_2 \delta_1 = 0, \quad \delta_1 \delta_2 + \delta_2 \delta_1 = 0.$$

Thus the associated Laplacian and fundamental solution of the heat equation decompose in the form:

$$\Delta_{\phi, g}^p = \bigoplus_{p=p_1+p_2} \Delta_{\phi_1, g_1}^{p_1} \otimes \text{Id} + \text{Id} \otimes \Delta_{\phi_2, g_2}^{p_2},$$

$$e^{-t\Delta_{\phi, g, B_a}^p} = \bigoplus_{p=p_1+p_2} e^{-t\Delta_{\phi_1, g_1}^{p_1}} \otimes e^{-t\Delta_{\phi_2, g_2, B_a}^{p_2}}.$$

Let  $f = f_1 f_2$  where  $f_i \in C^\infty(M_i)$ . We then have

$$\text{Tr}_{L^2}\{f e^{-t\Delta_{\phi, g, B_a}^p}\} = \sum_{p=p_1+p_2} \text{Tr}_{L^2}\{f_1 e^{-t\Delta_{\phi_1, g_1}^{p_1}}\} \cdot \text{Tr}_{L^2}\{f_2 e^{-t\Delta_{\phi_2, g_2, B_a}^{p_2}}\}.$$

Assertion (4) now follows by equating coefficients in the asymptotic expansion of the supertrace. □

### 3. Invariance theory

Let  $V$  be an  $m$  dimensional real vector space which is equipped with a positive definite inner product  $g(\cdot, \cdot)$ . Let  $O(V)$  be the associated orthogonal group. One says that a polynomial map  $f : \times^k V \rightarrow \mathbb{R}$  is an *orthogonal invariant* if

$$f(\xi v^1, \dots, \xi v^k) = f(v^1, \dots, v^k) \quad \forall \xi \in O(V) \text{ and } \forall (v^1, \dots, v^k) \in \times^k V.$$

Weyl’s first theorem of invariants [22, Theorem 2.9.A] is the following:

**Theorem 3.1.** *Every orthogonal invariant depending on  $k$  vectors  $(v_1, \dots, v_k)$  in  $\times^k V$  is expressible in terms of the  $k^2$  scalar invariants  $g(v_i, v_j)$ .*

Let  $\mathcal{I}_{k,m}$  be the set of all multilinear invariant maps from  $\times^k V$  to  $\mathbb{R}$ ; only the dimension  $m$  of  $V$  is really relevant so we suppress  $V$  from the notation. Given our interest is in  $O(V)$  and not  $SO(V)$  invariance, we have  $\mathcal{I}_{k,m} = \{0\}$  if  $k$  is odd. Consequently, we shall suppose that  $k$  is even henceforth. Let  $\Sigma_k$  be the group of all permutations of the set  $\{1, \dots, k\}$ . We define a multilinear invariant map  $p_{k,\sigma}$  for any permutation  $\sigma \in \Sigma_k$  by setting:

$$p_{k,\sigma}(v_1, \dots, v_k) := g(v_{\sigma(1)}, v_{\sigma(2)}) \cdots g(v_{\sigma(k-1)}, v_{\sigma(k)}).$$

**Theorem 3.2.**  $\mathcal{I}_{k,m} = \text{Span}_{\sigma \in \Sigma_k} \{p_{k,\sigma}\}$ .

**Proof.** We use Theorem 3.1 to express  $p \in \mathcal{I}_{k,m}$  in terms of monomials involving the inner products  $g(v_i, v_j)$ . Since  $p$  is multilinear:

$$p(cv_1, v_2, \dots, v_k) = cp(v_1, v_2, \dots, v_k).$$

Consequently, we need only consider monomials where the variable  $v_1$  appears exactly once as otherwise we contradict multilinearity. A similar observation holds for the remaining indices and these are exactly the expressions  $p_{k,\sigma}$  defined above. □



In view of **Theorem 3.2**, one says ‘invariant multilinear maps are given by contractions of indices’ as, relative to an orthonormal basis, the inner products involved correspond to contraction of indices in pairs. Let  $\{e_i\}$  be an orthonormal basis for the vector space  $V$  and let  $\omega = \omega_{i_1 i_2 \dots i_k} e_{i_1} \otimes \dots \otimes e_{i_k} \in \otimes^k V$ . We have, for example:

$$\mathcal{I}_{2,m} := \text{Span}\{\omega \rightarrow \omega_{ii}\}, \quad \mathcal{I}_{4,m} := \text{Span}\{\omega \rightarrow \omega_{ijj}, \omega \rightarrow \omega_{ijij}, \omega \rightarrow \omega_{ijji}\}.$$

Let  $\mathcal{P}_{n,m}$  be the space of invariant polynomials which are homogeneous of weight  $n$  in the derivatives of the metric tensor. Atiyah et al. [2] applied this formalism to study these spaces. In geodesic coordinate systems, all jets of the metric can be computed in terms of the covariant derivatives of the curvature tensor and vice versa. Thus, for example, if  $n = 4$ , an invariant  $P \in \mathcal{P}_{4,m}$  can be regarded as a map from a certain subspace

$$W \subset \{\otimes^6 T(M)\} \oplus \{\otimes^8 T(M)\}$$

to  $\mathbb{R}$  which is invariant under the action of the orthogonal group; here  $W$  is generated by the algebraic covariant derivatives  $\nabla^2 R \subset \otimes^6 T(M)$  and by the algebraic curvature tensors  $R \otimes R \subset \otimes^8 T(M)$ . As the subspace  $W$  is orthogonally invariant, extending  $P$  to be zero on  $W^\perp$  defines an orthogonally invariant map to which **Theorem 3.2** applies. Thus, for example, after taking into account the appropriate curvature symmetries, one has

$$\begin{aligned} \mathcal{P}_{2,m} &= \text{Span}\{\tau := R_{ijji}\}, \\ \mathcal{P}_{4,m} &= \text{Span}\{\tau^2, |\rho^2| := R_{ijk} R_{illk}, |R|^2 := R_{ijkl} R_{ijkl}, \Delta\tau := -R_{ijji;kk}\}. \end{aligned}$$

This analysis extends to form valued invariants with coefficients in an auxiliary vector bundle and gives rise to a heat equation proof of the index theorem for the classical elliptic complexes [2].

What is relevant to our analysis, however, is Weyl’s second main theorem [22, **Theorem 2.17.A**].

**Theorem 3.3.** *Every relation among scalar products is an algebraic consequence of the relation*

$$0 = \det \begin{pmatrix} g(v_1, w_1) & g(v_2, w_1) & \dots & g(v_{m+1}, w_1) \\ g(v_1, w_2) & g(v_2, w_2) & \dots & g(v_{m+1}, w_2) \\ \vdots & \vdots & \vdots & \vdots \\ g(v_1, w_{m+1}) & g(v_2, w_{m+1}) & \dots & g(v_{m+1}, w_{m+1}) \end{pmatrix}.$$

We remark that this relation can also be expressed in the form:

$$0 = g(v_1 \wedge \dots \wedge v_{m+1}, w_1 \wedge \dots \wedge w_{m+1}). \tag{3.1}$$

Let  $W$  be a vector space of dimension  $m - 1$ . Choose an inner product preserving inclusion  $i : W \subset V$  which embeds  $O(W) \subset O(V)$ . We define the restriction map

$$r : \mathcal{I}_{k,m} \rightarrow \mathcal{I}_{k,m-1}$$

which is characterized dually by the property:

$$r(p)(w_1, \dots, w_k) = p(i(w_1), \dots, i(w_k)).$$

If  $p$  is given by contractions of indices which range from 1 to  $m$ , then  $r(p)$  is given by restricting the range of summation to range from 1 to  $m - 1$ . Consequently, the map  $r$  is surjective. If  $k \geq 2m$  and if  $\sigma \in \Sigma_k$ , define:

$$\begin{aligned} \Theta_{k,m,\sigma}(v_1, \dots, v_k) &:= g(v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(m)}, v_{\sigma(m+1)} \wedge \dots \wedge v_{\sigma(2m)}) \\ &\quad \times g(v_{\sigma(2m+1)}, v_{\sigma(2m+2)}) \cdots g(v_{\sigma(k-1)}, v_{\sigma(k)}). \end{aligned}$$

**Theorem 3.4.** *Let  $m \geq 2$ .*

- (1)  $r : \mathcal{I}_{k,m} \rightarrow \mathcal{I}_{k,m-1}$  is surjective.
- (2)  $r : \mathcal{I}_{k,m} \rightarrow \mathcal{I}_{k,m-1}$  is injective if  $k < 2m$ .
- (3) If  $k \geq 2m$ , then  $\ker(r) \cap \mathcal{I}_{k,m} = \text{Span}_{\sigma \in \Sigma_k} \{ \Theta_{k,m,\sigma} \}$ .

**Proof.** We have already verified Assertion (1). To prove Assertion (2), we use [Theorem 3.2](#) to express  $p \in \mathcal{I}_{k,m}$  in terms of inner products. We use [Theorem 3.3](#), after making an appropriate dimension shift, to see that  $r(p)$  vanishes if and only if it can be written as sums of terms each of which is divisible by an appropriate determinant  $J$  of size  $m \times m$ . The desired result now follows from [Eq. \(3.1\)](#) and from the same arguments used to prove [Theorem 3.2](#). □

Previously we have considered invariants of the metric alone. The analysis extends easily to the twisted setting. We define

$$\text{weight}(\nabla^k \phi) = k \quad \text{and} \quad \text{weight}(\nabla^k R) = 2 + k.$$

Let  $\mathcal{Q}_{n,m}$  be the space of all  $O(m)$  invariant polynomials of total weight  $n$  in the components of  $R$ , the covariant derivatives of  $R$ , and the covariant derivatives of  $\phi$ . We do not admit  $\phi$  as a variable. Furthermore, we require that each monomial either does not involve the covariant derivatives of  $\phi$  at all or involves at least two covariant derivatives of  $\phi$ . We use the  $\mathbb{Z}_2$  action  $\phi \rightarrow -\phi$  to decompose

$$\mathcal{Q}_{n,m} = \mathcal{Q}_{n,m}^+ \oplus \mathcal{Q}_{n,m}^-, \quad \text{where} \quad \mathcal{Q}_{n,m}^\pm := \{ Q \in \mathcal{Q}_{n,m} : Q(\phi, g) = \pm Q(-\phi, g) \}.$$

The restriction map in [Theorem 3.4](#) induces natural surjective maps:

$$r : \mathcal{Q}_{n,m}^\pm \rightarrow \mathcal{Q}_{n,m-1}^\pm \rightarrow 0.$$

If  $(N, \phi_N, g_N)$  are structures in dimension  $m - 1$ , then we can define corresponding structures in dimension  $m$  by setting

$$(M, \phi_M, g_M) := (N \times S^1, \phi_N, g_N + d\theta^2).$$

If  $y \in \partial N$  is the point of evaluation, let  $(y, 1) \in \partial M$  be the corresponding point of evaluation—it does not matter which point is chosen on the circle owing to the rotational

symmetry. The restriction map  $r : \mathcal{Q}_{n,m} \rightarrow \mathcal{Q}_{n,m-1}$  is then characterized dually by the formula:

$$r(Q)(\phi_N, g_N)(y) = Q(\phi_N, g_N + d\theta^2)(y, 1).$$

**Lemma 3.5.**

- (1) If  $m$  is even, then  $a_{n,m}^{d+\delta}(\phi, g) \in \mathcal{Q}_{n,m}^+ \cap \ker r$ .
- (2) If  $m$  is odd, then  $a_{n,m}^{d+\delta}(\phi, g) \in \mathcal{Q}_{n,m}^- \cap \ker r$ .

**Proof.** Standard arguments [12] show the invariants  $a_{n,m}^{d+\delta}(\phi, g)$  are homogeneous of weight  $n$  in the jets of the metric and of  $\phi$ . Let  $\nabla$  be the Levi-Civita connection on  $AM$ . By Lemma 2.2 (1):

$$\Delta_{\phi,g}^p = \Delta_g + \frac{1}{2} \gamma_i \gamma_j \Omega_{ij} + \phi_{;i} \phi_{;i} - \phi_{;ji} (\epsilon_i i_j - i_j \epsilon_i).$$

Thus the undifferentiated variable  $\phi$  does not play a role in these invariants. Furthermore, either at least two covariant derivatives of  $\phi$  appear or only the curvature  $R$  appears in each Weyl monomial of  $a_{n,m}^{d+\delta}(\phi, g)$ . This shows that

$$a_{n,m}^{d+\delta}(\phi, g) \in \mathcal{Q}_{n,m}.$$

We use Lemma 2.3 (2) to see that  $a_{n,m}^{d+\delta}(\phi, g)$  is an odd function of  $\phi$  if  $m$  is odd and an even function of  $\phi$  if  $m$  is even. To complete the proof, we must show  $ra_{n,m}^{d+\delta} = 0$ . Suppose that  $M = N \times S^1$  has the product metric and that  $\phi = \phi_N$  is independent of the angular parameter  $\theta \in S^1$ . As  $\phi_{S^1} = 0$ , we use Lemma 2.3 (3) to see  $a_{n,1}^{d+\delta}(0, g_{S^1}) = (-1)^1 a_{n,1}^{d+\delta}(0, g_{S^1}) = 0$  for all  $n$ . Thus Lemma 2.3 (4a) implies that  $a_{n,m}^{d+\delta}(\phi_M, g_N) = 0$ . This shows that  $ra_{n,m}^{d+\delta} = 0$ . □

Assertion (1) of Lemma 1.3 will follow from the following result.

**Lemma 3.6.** If  $m$  is odd, then  $\mathcal{Q}_{m+1,m}^- \cap \ker r = \text{Span}\{\mathcal{E}_{m+1,m}\}$ .

**Proof.** Let  $0 \neq Q \in \mathcal{Q}_{m+1,m}^-$ . Let  $A$  be a monomial of  $Q$  of the form:

$$A = \phi_{;\alpha_1} \cdots \phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1; \beta_1} \cdots R_{i_v j_v k_v \ell_v; \beta_v}$$

where  $\alpha_\mu$  and  $\beta_\nu$  denote appropriate collections of indices. Then

$$m + 1 = \text{weight}(A) = \sum_{\mu} |\alpha_\mu| + \sum_{\nu} (2 + |\beta_\nu|).$$

By definition, the empty sum is 0. Thus  $\sum_{\mu}$  is to be ignored if  $u = 0$  and  $\sum_{\nu}$  is to be ignored if  $v = 0$ . Let  $k$  be total number of indices present in  $A$ ;

$$k := \sum_{\mu} |\alpha_\mu| + \sum_{\nu} (4 + |\beta_\nu|) = \text{weight}(A) + 2v = m + 1 + 2v.$$

We apply Weyl’s second main theorem of invariance theory as discussed above. To ensure that  $rQ = 0$ , we must contract  $2m$  indices in  $A$  using the  $\varepsilon$  tensor and then contract the remaining indices of  $A$  in pairs. Consequently, at least  $2m$  indices must appear in  $A$  so

$$2m \leq k = m + 1 + 2v = 2m + 2 - \sum_{\mu} |\alpha_{\mu}| - \sum_v |\beta_v| \leq 2m + 2. \tag{3.2}$$

Since  $m$  is odd,  $2m$ ,  $m + 1 + 2v$ , and  $2m + 2$  are all even. Thus only one of the two inequalities given in Display (3.2) can be strict. As  $Q(-\phi, g) = -Q(\phi, g)$ ,  $u$  must be *odd*. Thus

$$\sum_{\mu} |\alpha_{\mu}| > 0$$

so the second inequality in Eq. (3.2) is strict. Thus exactly  $2m = k$  indices appear in  $A$  and all are contracted using the  $\varepsilon$  tensor. The first and second Bianchi identity show  $R_{****;\beta} = 0$  if 3 indices are alternated. Thus at most two  $i$  indices and at most two  $j$  indices can appear in each  $R_{****;\beta}$  variable. This shows that

$$|\beta_v| = 0 \quad \text{for all } v.$$

Furthermore, the two possibilities are  $R_{i_1 i_2 j_2 j_1}$  or  $R_{i_1 j_1 i_2 j_2}$ . The first Bianchi identity can then be used to express the second variable in terms of the first. Since

$$u \leq |\alpha_1| + \dots + |\alpha_u| = 2$$

and  $u$  is odd,  $u = 1$  and  $|\alpha_1| = 2$ , since either 0 or at least two covariant derivatives of  $\phi$  appear in each monomial of  $Q \in \mathcal{Q}_{n,m}$ . Thus we are in fact dealing with a multiple of  $\mathcal{E}_{m+1,m} := \varepsilon^J \phi_{;i_1 j_1} \mathcal{R}_{j_2}^{L,m}$ . □

To complete the proof of Lemma 1.3, we study the boundary invariants. Let  $\tilde{\nabla}$  denote the Levi–Civita connection of the boundary. We consider polynomials in the components of the tensors

$$\{R, \nabla R, \nabla^2 R, \dots, L, \tilde{\nabla} L, \tilde{\nabla}^2 L, \dots, \nabla \phi, \nabla^2 \phi, \dots\}.$$

Again, we do not introduce the variable  $\phi$ . We let

$$\text{weight}(\nabla^k R) := 2 + k, \text{weight}(\tilde{\nabla}^k L) := 1 + k, \quad \text{and} \quad \text{weight}(\nabla^k \phi) = k.$$

Let  $\tilde{\mathcal{Q}}_{n,m}$  be the space of all  $O(m - 1)$  invariant polynomials of total weight  $n$  where we admit monomials which either do not involve the covariant derivatives of  $\phi$  at all or which involve at least two covariant derivatives of  $\phi$ .

Let  $\tilde{\mathcal{P}}_{n,m} \subset \tilde{\mathcal{Q}}_{n,m}$  be the subspace of invariants which do not involve the covariant derivatives of  $\phi$ . Setting  $\phi = 0$  defines a natural map from  $\tilde{\mathcal{Q}}_{n,m}$  to  $\tilde{\mathcal{P}}_{n,m}$ . If  $P \in \tilde{\mathcal{P}}_{n,m}$ , then the evaluation  $\mathfrak{I}(P)(g) \in \mathbb{R}$  is defined by setting:

$$\mathfrak{I}P(g) := \int_{\partial M} P(g)(y) \, dy.$$

By Lemma 2.3 (3),  $\mathcal{J}a_{m+1,m,0}^{d+\delta}(0, g) = 0$ . The same argument as that given to establish Lemma 3.5 can be used to show that  $a_{n,m,k}^{d+\delta} \in \tilde{\mathcal{Q}}_{n-k-1} \cap \ker r$ . The remaining assertions of Lemma 1.3 will now follow from the following result.

**Lemma 3.7.**

- (1)  $\tilde{\mathcal{Q}}_{n,m} \cap \ker r = \{0\}$  if  $n < m - 1$ .
- (2)  $\tilde{\mathcal{Q}}_{m-1,m} \cap \ker r = \text{Span}_k \{ \mathcal{F}_{m-1,m}^k \}$ .
- (3)  $\tilde{\mathcal{Q}}_{m,m} \cap \ker r = \text{Span}_k \{ \mathcal{F}_{m,m}^{1,k}, \mathcal{F}_{m,m}^{2,k} \} + \{ \tilde{\mathcal{P}}_{m,m} \cap \ker r \}$ .
- (4)  $\tilde{\mathcal{P}}_{m,m} \cap \ker r \cap \ker \mathcal{J} = \text{Span}_k \{ \mathcal{F}_{m,m}^{3,k} \}$ .

**Proof.** Let  $0 \neq Q \in \tilde{\mathcal{Q}}_{n,m} \cap \ker r$  and let  $A$  be a monomial of  $Q$  of weight  $n$  where:

$$A := \phi_{;\alpha_1} \cdots \phi_{;\alpha_u} R_{i_1 j_1 k_1 \ell_1; \beta_1} \cdots R_{i_v j_v k_v \ell_v; \beta_v} L_{a_1 b_1; \gamma_1} \cdots L_{a_w b_w; \gamma_w},$$

$$n := \sum_{\mu} |\alpha_{\mu}| + \sum_{\nu} (|\beta_{\nu}| + 2) + \sum_{\sigma} (|\gamma_{\sigma}| + 1).$$

To ensure that  $rQ = 0$ , we contract  $2(m - 1)$  tangential indices in  $A$  using the  $\varepsilon$  tensor; the remaining tangential indices must be contracted in pairs. Since the structure group is  $O(m - 1)$ , the normal index ‘ $m$ ’ can stand alone and unchanged. Let  $k_T$  be the total number of tangential indices in  $A$ , and let  $k_m$  be the total number of times the normal index  $m$  appears in  $A$ . We estimate:

$$2m - 2 \leq k_T \leq k_T + k_m = \sum_{\mu} |\alpha_{\mu}| + \sum_{\nu} (|\beta_{\nu}| + 4) + \sum_{\sigma} (|\gamma_{\sigma}| + 2) = n + 2v + w$$

$$= 2n - \sum_{\mu} |\alpha_{\mu}| - \sum_{\nu} |\beta_{\nu}| - \sum_{\sigma} |\gamma_{\sigma}| \leq 2n. \tag{3.3}$$

Assertion (1) of the lemma follows as this is not possible if  $n < m - 1$ .

We set  $n = m - 1$  to prove Assertion (2). All the inequalities of Display (3.3) must have been equalities so there are no covariant derivatives and thus the  $\phi$  variables do not appear. All the indices are tangential and are contracted using the  $\varepsilon$  tensor. After using the first Bianchi identity, we see that this leads to the invariants  $\mathcal{F}_{m-1,m}^k$  which proves Assertion (2).

Let  $n = m$ . Display (3.3) involves a total increase of 2. Thus at most two explicit covariant derivatives are present. However, unless at least two covariant derivatives are present,  $\phi$  is not involved and this leads to invariants in  $\tilde{\mathcal{P}}_{m,m} \cap \ker r$ . Thus we may suppose exactly two explicit covariant derivatives are present—and all of them appear on  $\phi$ . Consequently

$$k_T = 2m - 2, \quad k_m = 0, \quad \sum_{\mu} |\alpha_{\mu}| = 2, \quad \sum_{\nu} |\beta_{\nu}| = 0, \quad \text{and} \quad \sum_{\sigma} |\gamma_{\sigma}| = 0.$$

Since every index is tangential and all are contracted using the tensor  $\varepsilon$ , after applying the Bianchi identities, we obtain the invariants  $\mathcal{F}_{m,m}^{1,k}$  and  $\mathcal{F}_{m,m}^{2,k}$ . This completes the proof of Assertion (3).

To prove Assertion (4), we set  $\phi = 0$  and consider only metric invariants. Let  $\tilde{\mathcal{P}}_{n,m}^p$  be the space of  $p$  form valued invariants which are homogeneous of degree  $n$  in the derivatives of the metric;  $\tilde{\mathcal{P}}_{n,m} = \tilde{\mathcal{P}}_{n,m}^0$ .

Let  $\tilde{\delta} : \tilde{\mathcal{P}}_{n,m}^p \rightarrow \tilde{\mathcal{P}}_{n+1,m}^{p-1}$  be the coderivative of the boundary. Results of [11] describe the cohomology groups of this complex. When combined with standard methods of invariance theory they yield the following observations:

- (1)  $r$  is a surjective map from  $\tilde{\mathcal{P}}_{n,m}^p$  to  $\tilde{\mathcal{P}}_{n,m-1}^p$  with  $r\tilde{\delta} = \tilde{\delta}r$ .
- (2) If  $n \neq m - 1$ , then  $\tilde{\mathcal{P}}_{n,m}^0 \cap \ker \mathcal{J} = \tilde{\delta}\tilde{\mathcal{P}}_{n-1,m}^1$ .
- (3) If  $n \neq m - 1$ , then  $\tilde{\mathcal{P}}_{n-1,m}^1 \cap \ker \tilde{\delta} = \tilde{\delta}\tilde{\mathcal{P}}_{n-2,m}^2$ .

Let  $P_{m,m} \in \tilde{\mathcal{P}}_{m,m} \cap \ker r \cap \ker \mathcal{J}$ . Choose  $P_{m-1,m}^1 \in \tilde{\mathcal{P}}_{m-1,m}^1$  so  $\tilde{\delta}P_{m-1,m}^1 = P_{m,m}$ . Unfortunately,  $rP_{m-1,m}^1$  need not vanish and we must adjust  $P_{m-1,m}^1$ . Since

$$\tilde{\delta}rP_{m-1,m}^1 = r\tilde{\delta}P_{m-1,m}^1 = rP_{m,m} = 0,$$

we may choose  $P_{m-2,m-1}^2 \in \tilde{\mathcal{P}}_{m-2,m-1}^2$  so  $\tilde{\delta}P_{m-2,m-1}^2 = rP_{m-1,m}^1$ . Since  $r$  is surjective, we may choose  $P_{m-2,m}^2 \in \tilde{\mathcal{P}}_{m-2,m}^2$  so  $rP_{m-2,m}^2 = P_{m-2,m-1}^2$ . Then

$$\begin{aligned} \tilde{\delta}\{P_{m-1,m}^1 - \tilde{\delta}P_{m-2,m}^2\} &= \tilde{\delta}P_{m-1,m}^1 = P_{m,m}, \\ r\{P_{m-1,m}^1 - \tilde{\delta}P_{m-2,m}^2\} &= rP_{m-1,m}^1 - \tilde{\delta}rP_{m-2,m}^2 = rP_{m-1,m}^1 - \tilde{\delta}P_{m-2,m-1}^2 = 0. \end{aligned}$$

Consequently

$$\tilde{\mathcal{P}}_{m,m} \cap \ker r \cap \ker \mathcal{J} = \tilde{\delta}\{\tilde{\mathcal{P}}_{m-1,m}^1 \cap \ker r\}. \tag{3.4}$$

Let  $0 \neq P_{m-1,m}^1 \in \tilde{\mathcal{P}}_{m-1,m}^1 \cap \ker r$  and let

$$A = R_{i_1 j_1 k_1 \ell_1; \beta_1} \cdots R_{i_v j_v k_v \ell_v; \beta_v} L_{a_1 b_1; \gamma_1} \cdots L_{a_w b_w; \gamma_w} e^c$$

be a monomial of  $P_{m-1,m}^1$ . Since  $rP_{m-1,m}^1 = 0$ , we must contract  $2(m - 1)$  indices in  $A$  using the  $\varepsilon$  tensor and contract the remaining indices in pairs. We estimate

$$\begin{aligned} 2(m - 1) \leq k_T \leq k_T + k_m &= \sum_v (|\beta_v| + 4) + \sum_\sigma (|\gamma_\sigma| + 2) + 1 \\ &= m - 1 + 2v + w + 1 \\ &= 2(m - 1) - \sum_v |\beta_v| - \sum_\sigma |\gamma_\sigma| + 1 \leq 2(m - 1) + 1. \end{aligned} \tag{3.5}$$

This sequence of inequalities represents a total increase of 1. Thus  $k_T = 2(m - 1)$  and every tangential index is contracted using the  $\varepsilon$  tensor. We have

$$L_{c_2 c_3; c_1} - L_{c_1 c_3; c_2} = R_{c_1 c_2 c_3 m}. \tag{3.6}$$

We may therefore assume  $|\gamma_\sigma| = 0$  so there are no tangential derivatives of  $L$  present. If  $k_m = 0$ , then every index is contracted using the  $\varepsilon$  tensor. Thus the Bianchi identities show

$|\beta_\nu| = 0$  for all  $\nu$ . This means that every inequality in Display (3.5) is an equality which is impossible. Consequently  $k_m = 1$  and  $\sum_\nu |\beta_\nu| = 0$ . This leads to the invariants

$$\mathcal{G}_{m-1,m}^k := \varepsilon_A^B \mathcal{R}_{B,1}^{A,2k} R_{a_{2k+1}a_{2k+2}mb_{2k+1}} \mathcal{L}_{B,2k+3}^{A,m-1} e^{b_{2k+2}}.$$

Assertion (4) now follows from Eq. (3.4) since  $\tilde{\delta} \mathcal{G}_{m-1,m}^k = -\mathcal{F}_{m,m}^{3,k}$ . □

#### 4. Product formulas, special case computations, and functorial properties

Throughout this section, we adopt the notation of Lemma 1.3. We begin with a result which is based on product formulas.

##### Lemma 4.1.

- (1) If  $m = 2\bar{m} + 1$ , then  $c_{m+1,m} = (1/\sqrt{\pi} 8^{\bar{m}} \pi^{\bar{m}} \bar{m}!)$ .
- (2) If  $k > 0$ , then  $c_{m+1,m,1}^k = (1/\pi^k 8^k k!) c_{m-2k+1,m-2k,1}^0$ .
- (3) If  $k > 0$ , then  $c_{m+1,m,0}^{i,k} = (1/\pi^k 8^k k!) c_{m-2k+1,m-2k,0}^{i,0}$ .
- (4) We have  $c_{m+1,m,0}^{1,0} = (1/\sqrt{\pi})(1/(m-2)! \text{vol}(S^{m-2}))$  and  $c_{m+1,m,0}^{2,0} = 0$ .

**Proof.** Give  $S^m$  and  $D^m$  the standard metrics  $g_{S,m}$  and  $g_{D,m}$ . We then have

$$\varepsilon_J^I \mathcal{R}_{J,1}^{I,2\bar{m}}(g_{S,m}) = 2^{\bar{m}}(2\bar{m})! \quad \text{and} \quad \varepsilon_A^B \mathcal{L}_{A,1}^{B,m-1}(g_{D,m}) = (m-1)! \tag{4.1}$$

Let  $m = 2\bar{m} + 1$ . Give  $M := S^1 \times S^{2\bar{m}}$  the product structures where  $\phi_2 = 0$ . By Theorem 1.1

(1) and Lemma 2.3 (4) we have

$$a_{m+1,m}^{d+\delta}(\phi_M, g_M) = a_{2,1}^{d+\delta}(\phi_1, g_{S,1}) a_{2\bar{m},2\bar{m}}(0, g_{S,2\bar{m}}).$$

Consequently, by Eq. (4.1) and by Theorem 1.1 (4):

$$\begin{aligned} a_{m+1,m}^{d+\delta}(\phi, g) &= c_{m+1,m} \phi_{;11} 2^{\bar{m}}(2\bar{m})! = a_{2,1}^{d+\delta}(\phi_1, g_{S,1}) \cdot a_{2\bar{m},2\bar{m}}^{d+\delta}(0, g_{S,2\bar{m}}) \\ &= \frac{1}{\sqrt{\pi}} \phi_{;11} \frac{1}{8^{\bar{m}} \pi^{\bar{m}} \bar{m}!} 2^{\bar{m}}(2\bar{m})!. \end{aligned}$$

We complete the proof of Asssertion (1) by using this relation to solve for  $c_{m+1,m}$ :

$$c_{m+1,m} = \frac{1}{\sqrt{\pi} 8^{\bar{m}} \pi^{\bar{m}} \bar{m}!}.$$

Fix  $k > 0$ . Give  $M = S^{2k} \times D^{m-2k}$  the product structures where  $\phi_1 = 0$ . We argue as in the proof of Asssertion (1) to see that:

$$\begin{aligned} a_{m+1,m,1}^{d+\delta}(\phi_M, g_M) &= \sum_j c_{m+1,m,1}^j \mathcal{F}_{m,m}^j(\phi_M, g_M) = c_{m+1,m,1}^k 2^k(2k)! \cdot (m-2k-1)! \\ &= a_{2k,2k}^{d+\delta}(0, g_{S,2k}) \cdot a_{m-2k+1,m-2k,1}^{d+\delta}(0, g_{D,m-2k}) \\ &= \frac{1}{\pi^k 8^k k!} 2^k(2k)! c_{2k+1,2k,1}^0 (m-2k-1)!. \end{aligned}$$

This equation relates  $c_{m+1,m,1}^k$  and  $c_{m+1,m,1}^0$  and thereby establishes Assertion (2); the proof of Assertion (3) is similar.

Let  $M := S^1 \times D^{m-1}$  where  $\phi = \phi(\theta)$  depends only on  $S^1$ . We use Theorem 1.1 to determine  $a_{m-1,m-1,0}^{d+\delta}(0, g_{D,m-1})$ . As  $a_{2,1}^{d+\delta} = (1/\sqrt{\pi})\phi_{;11}$ , we argue as above to see

$$\begin{aligned} a_{m+1,m,0}^{d+\delta}(\phi, g) &= \{c_{m+1,m,0}^{1,0}\phi_{;11} + c_{m+1,m,0}^{2,0}\phi_{;1}\phi_{;1}\}(m-2)! \\ &= a_{2,1}^{d+\delta}(\phi, d\theta^2) \cdot a_{m-1,m-1,0}^{d+\delta}(0, g_{D^{m-1}}) = \frac{1}{\sqrt{\pi}}\phi_{;11} \frac{(m-2)!}{\text{vol}(S^{m-2})(m-2)!}. \end{aligned}$$

We solve for  $c_{m+1,m,0}^{1,0}$  and  $c_{m+1,m,0}^{2,0}$  to establish Assertion (4). □

By Lemma 4.1, we need only determine  $c_{m+1,m,1}^0$  and  $c_{m+1,m,0}^{3,0}$  to complete the proof of Theorem 1.2. As these terms do not involve  $\phi$ , we set  $\phi = 0$  henceforth. We introduce universal constants  $\bar{c}_{n,m,k}^v$  so that if  $\mathcal{B}$  defines mixed boundary conditions for an operator of Laplace type, then the heat trace asymptotics have the form:

$$a_{n,m,k}(y, D, \mathcal{B}) = \bar{c}_{n,m,k}^0 \text{Tr}\{S^{n-k-1}\} + \bar{c}_{n,m,k}^3 \text{Tr}\{E_{;m}S^{n-k-4}\} + \dots$$

We will use the method of universal examples to show that only  $\text{Tr}\{S^{m-1}\}$  is relevant in computing  $\{a_{m,m,0}^{d+\delta}(0, g), a_{m+1,m,1}^{d+\delta}(0, g)\}$  and that only  $\text{Tr}\{E_{;m}S^{m-3}\}$  is relevant in computing  $a_{m+1,m,0}^{d+\delta}(0, g)$ . This will enable us to show the following lemma.

**Lemma 4.2.**

- (1) If  $m \geq 2$ , then  $c_{m+1,m,1}^0 = \bar{c}_{m+1,m,1}^0$  and  $\bar{c}_{m,m,0}^0 = (1/(m-1)!\text{vol}(S^{m-1}))$ .
- (2)  $c_{4,3,0}^{3,0} = 0$ . If  $m > 3$ , then  $c_{m+1,m,0}^{3,0} = \bar{c}_{m+1,m,0}^3$ .

**Remark.** The constants  $\bar{c}_{n,m,k}^0$  and  $\bar{c}_{n,m,k}^3$  have been determined in [5]; after a bit of work converting from  $\Gamma$  functions into volumes of spheres one checks the value of  $\bar{c}_{m+1,m,1}^0$  given here is consistent with the value given in [5]; this provides a valuable check on our methodology.

**Proof.** We shall prove Lemma 4.2 by making a special case calculation. Let  $m \geq 2$ . To simplify the notation, let

$$P_m(g) := a_{m+1,m,1}^{d+\delta}(0, g), \quad c_m := c_{m+1,m,1}^0, \quad \bar{c}_m := \bar{c}_{m+1,m,1}^0.$$

Let  $(y_1, \dots, y_{m-1})$  be the usual coordinates on  $\mathbb{R}^{m-1}$ . Let  $f(y)$  be a smooth even function of  $y$  and let

$$M_m := \{(y, r) \in \mathbb{R}^m : r \geq f(y)\}.$$

Let  $\{A_1, \dots, A_{m-1}\}$  be distinct real constants. We choose  $f$  so that

$$f(0) = 0, \quad (\partial_i^y f)(0) = 0, \quad \text{and} \quad (\partial_i^y \partial_j^y f)(0) = A_i \delta_{ij}. \tag{4.2}$$



Give  $\mathbb{R}^m$  the usual flat metric. Then  $L_{ij}(0) = -A_i\delta_{ij}$ . We use Lemma 1.3 to compute:

$$P_m(g)(0) = (m - 1)!c_m\mathcal{A}, \quad \text{where } \mathcal{A} := (-1)^{m-1}A_1 \cdots A_{m-1}. \tag{4.3}$$

Because  $R = 0$ , we have  $E = 0$  and  $\Omega = 0$ . Thus there exists a polynomial  $Q_m$  of total weight  $m - 1$  in the tangential covariant derivatives of  $\{\chi, L, S\}$  so that

$$P_m = \sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^m)}\{Q_m(\cdot)\}.$$

We must control  $\tilde{\nabla}^k L$  for  $k \geq 1$ . Since the curvature of  $\mathbb{R}^m$  vanishes, Eq. (3.6) shows that  $\tilde{\nabla} L$  is a totally symmetric tensor field. Since  $f$  is an even function,  $\tilde{\nabla}^k L$  vanishes at the origin if  $k$  is odd. For  $k$  even, the components of  $\tilde{\nabla}^k L(0)$  are polynomials in the derivatives of the defining function  $f$ . Let  $\mathfrak{K}$  denote the ideal in the algebra of all polynomials in the jets of  $f$  which is generated by the monomials  $\{A_1^2, \dots, A_{m-1}^2\}$ . In light of Eq. (4.3), we shall work modulo  $\mathfrak{K}$  since such elements can not contribute to  $\mathcal{A}$ .

We first study  $\tilde{\nabla}^2 L$ . This is not a symmetric tensor field. Let  $\tilde{R}$  be the curvature of the Levi–Civita connection of  $\partial M$ . Let  $\{e_1, \dots, e_{m-1}\}$  be an orthonormal frame for the tangent bundle of the boundary so that  $e_i(0) = \partial_i^y$ . Then

$$\begin{aligned} \tilde{R}_{b_1 b_2 b_3 b_4} &= L_{b_1 b_4} L_{b_2 b_3} - L_{b_1 b_3} L_{b_2 b_4}, \quad \text{and} \\ L_{a_1 a_2 : a_3 a_4} - L_{a_1 a_2 : a_4 a_3} &= \tilde{R}_{a_3 a_4 a_1 a_5} L_{a_5 a_2} + \tilde{R}_{a_3 a_4 a_2 a_5} L_{a_5 a_1}. \end{aligned}$$

This shows that  $A_{a_5}^2$  divides  $\{\tilde{R}_{a_3 a_4 a_1 a_5} L_{a_5 a_2} + \tilde{R}_{a_3 a_4 a_2 a_5} L_{a_5 a_1}\}(0)$ . Consequently  $\tilde{\nabla}^2 L(0)$  is totally symmetric modulo the ideal  $\mathfrak{K}$ . Since the components of  $\tilde{\nabla}^2 L$  are linear in the four jets of  $f$  and quadratic in the two jets of  $f$ , we may choose the four jets of  $f$  to kill the symmetrization of  $(\tilde{\nabla}^2 L)(0)$  and thereby ensure  $(\tilde{\nabla}^2 L)(0) \in \mathfrak{K}$ . Similarly, by choosing  $\tilde{\nabla}^{k+2} f(0)$  appropriately, we may suppose that

$$(\tilde{\nabla}^k L)(0) \in \mathfrak{K} \quad \text{for } k > 0.$$

We therefore suppress  $\tilde{\nabla}^k L$  henceforth in the proof of Assertions (1) and (2). By Lemma 2.2 (5),  $\chi_{:a} = 2L_{ab}(\epsilon_b \dot{m} + \epsilon_m \dot{b})$ . Thus further covariant differentiation of  $\chi$  only involves covariantly differentiating  $\epsilon_b \dot{m} + \epsilon_m \dot{b}$ . Thus inductively there exist suitably chosen endomorphisms  $\mathcal{E}_\star$  of weight 0, so

$$\chi_{:a_1 \cdots a_k} = L_{a_1 b_1} L_{a_2 b_2} \cdots L_{a_k b_k} \mathcal{E}_{b_1 \cdots b_k}. \tag{4.4}$$

If a  $\chi_{:a_1 \cdots}$  term appears, we must contract it with another index  $a_1$ ; Eq. (4.4) contains no  $L_{a_1 a_1}$  term. Consequently this contraction involves a different variable which produces an  $A_{a_1}^2$  term; such terms can be ignored in light of Eq. (4.3). Similarly since

$$S = -L_{ab}\epsilon_b \dot{a} \text{ on } \Lambda(\mathbb{R}^{m-1}) \quad \text{and} \quad S = 0 \text{ on } \Lambda(\mathbb{R}^{m-1}) \wedge dr,$$

$\tilde{\nabla}^k S$  plays no role if  $k \geq 1$ . If a  $L_{a_1 b_1}$  term appears where  $a_1$  is not to be contracted with  $b_1$ , then  $A$  must be divisible by  $A_{a_1}^2$ . If the term  $L_{aa}$  appears in a monomial  $Q$ , then we may factor  $Q = L_{aa}Q_0$  and then apply Lemma 3.7 (1) to see the supertrace of  $Q_0$  vanishes.

Thus  $L$  does not appear as a variable. This shows that only the monomial  $S^{m-1}$  is relevant. Consequently

$$P_m(g)(0) = \bar{c}_m \sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^m)} \{S^{m-1}\}(0).$$

Since  $S$  is zero on  $\Lambda^p(\mathbb{R}^{m-1}) \wedge dr$ ,

$$P_m(g)(0) = \bar{c}_m \sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^{m-1})} \{S^{m-1}\}(0). \tag{4.5}$$

We may decompose

$$\begin{aligned} \Lambda(\mathbb{R}^{m-1}) &= \Lambda(\mathbb{R}) \otimes \cdots \otimes \Lambda(\mathbb{R}) \quad \text{and} \\ S &= \sum_{1 \leq i \leq m-1} \text{Id} \otimes \cdots \otimes \text{Id} \otimes S_i \otimes \text{Id} \otimes \cdots \otimes \text{Id}, \quad \text{where} \\ S_i &= 0 \text{ on } \Lambda^0(\mathbb{R}) \text{ and } S_i = -A_i \text{ on } \Lambda^1(\mathbb{R}). \end{aligned}$$

The supertrace of  $\text{Id}$  is zero. Furthermore, the supertrace of the tensor product is the product of the supertraces. Thus only  $(m-1)! S_1 \otimes \cdots \otimes S_{m-1}$  survives in the supertrace of  $S^{m-1}$ . Since the supertrace of  $S_i$  is  $-A_i$ , we have that:

$$\sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^{m-1})} \{S^{m-1}\} = (m-1)!A. \tag{4.6}$$

Assertion (1) part one now follows from Eqs. (4.3), (4.5) and (4.6).

The invariant  $a_{m,m,0}^{d+\delta}$  is homogeneous of weight  $m-1$  and is in the kernel of  $r$ . Thus we can use exactly the same line of argument to show:

$$a_{m,m,0}^{d+\delta}(0, g)(0) = (m-1)!A \bar{c}_{m,m,0}^{-0}.$$

We use [Theorem 1.1](#) to evaluate  $a_{m,m,0}^{d+\delta}(0, g)(0)$  and establish Assertion (1) part two.

The proof of Assertion (2) is similar. Let  $m \geq 3$ . To simplify the notation, set

$$P_{m+1}(g) := a_{m+1,m,0}^{d+\delta}(0, g), \quad c_{m+1} := c_{m+1,m,0}^{3,0}, \quad \text{and} \quad \bar{c}_{m+1} := \bar{c}_{m+1,m,0}^{-3}.$$

Let  $(u_1, u_2, y_1, \dots, y_{m-3}, r)$  be coordinates on  $\mathbb{R}^m$ . Let  $f(y)$  satisfy the normalizations of [Eq. \(4.2\)](#). We set  $M = \{x \in \mathbb{R}^m : r \geq f(y)\}$  and

$$ds_M^2 := du_1^2 + e^{-A_0 u_1^2 r} du_2^2 + dy_1^2 + \cdots + dy_{m-3}^2 + dr^2.$$

Then  $R(\cdot)(0) = 0$  and the non-vanishing components of  $L$  and  $\nabla R$  at the origin are given, up to the usual  $\mathbb{Z}_2$  symmetries, by

$$L(\partial_i^y, \partial_j^y)(0) = -A_i \delta_{ij}, \quad \text{and} \quad R(\partial_1^u, \partial_2^u, \partial_2^u, \partial_1^u; \partial_r) = R(\partial_1^u, \partial_2^u, \partial_2^u, \partial_r; \partial_1^u) = A_0.$$

Let  $\mathcal{A} = (-1)^{m-3} A_0 A_1 \cdots A_{m-3}$ . We apply [Lemma 1.3](#) to see

$$P_{m+1}(g)(0) = 2(m-3)!c_{m+1}\mathcal{A}. \tag{4.7}$$

We now let  $\mathfrak{K}$  be the ideal generated by the elements  $\{A_0^2, A_1^2, \dots, A_{m-3}^2\}$ . If we set  $A_0 = 0$ , then the manifold is a product of the manifold considered previously with a flat factor. This shows that  $\nabla^k R(0), \nabla^k E(0), \nabla^k \Omega(0)$  are all divisible by  $A_0$  for  $k \geq 1$  and vanish if  $k = 0$ .

We consider terms which can give rise to  $\mathcal{A}$  after taking the supertrace. Let  $\mathcal{E}$  denote a generic polynomial in the tangential covariant derivatives of  $L$ , of  $S$ , and of  $\chi$  when  $A_0$  is set to zero. Since we are not interested in terms which are divisible by  $A_0^2$  and since  $A_0$  has weight 3, we factor out a term which can be linear in  $A_0$  to express  $P_{m+1}$  symbolically as

$$P_{m+1} = \sum_p (-1)^p \text{Tr}_{\Lambda^p(M)} \left\{ \sum_{k \geq 1} \nabla^k R \cdot \mathcal{E}_{m-k-2}^R + \sum_{k \geq 1} \nabla^k E \cdot \mathcal{E}_{m-k-2}^E \right. \\ \left. + \sum_{k \geq 1} \nabla^k \Omega \cdot \mathcal{E}_{m-k-2}^\Omega + \sum_{k \geq 2} \tilde{\nabla}^k L \cdot \mathcal{E}_{m-k-1}^L \right. \\ \left. + \sum_{k \geq 2} \tilde{\nabla}^k S \cdot \mathcal{E}_{m-k-1}^S + \sum_{k \geq 3} \tilde{\nabla}^k \chi \cdot \mathcal{E}_{m-k}^\chi \right\}.$$

We set  $A_0 = 0$  in studying the ‘coefficient’ monomials  $\mathcal{E}$ . Thus the arguments given above in the proof of Assertion (1) shows only powers of  $S$  are relevant so

$$P_{m+1} = \sum_p (-1)^p \text{Tr}_{\Lambda^p(M)} \\ \times \left\{ \sum_{k \geq 1} \nabla^k R \cdot S^{m-k-2} + \sum_{k \geq 1} \nabla^k E \cdot S^{m-k-2} + \sum_{k \geq 1} \nabla^k \Omega \cdot S^{m-k-2} \right. \\ \left. + \sum_{k \geq 2} \tilde{\nabla}^k L \cdot S^{m-k-1} + \sum_{k \geq 2} \tilde{\nabla}^k S \cdot S^{m-k-1} + \sum_{k \geq 3} \tilde{\nabla}^k \chi \cdot S^{m-k} \right\}. \tag{4.8}$$

By Lemma 3.7:

$$\sum_p (-1)^p \text{Tr}_{\Lambda^p(M)} \{S^k\} = 0 \quad \text{for } k < m - 1. \tag{4.9}$$

Thus the terms in  $\nabla^k R$  and  $\tilde{\nabla}^k L$  do not appear in Eq. (4.8) since, being scalars, they could be moved outside Tr. As  $\Omega$  is skew-adjoint and  $S$  is self-adjoint, this term does not appear. Terms involving  $\tilde{\nabla}^k S$  must be fully contracted and, modulo lower order terms which can be absorbed at an earlier stage, have the form:

$$S_{:a_1 a_1 a_2 a_2 \dots} S^k = \frac{1}{k+1} \{S^{k+1}\}_{:a_1 a_1 a_2 a_2 \dots} + O(A_0^2).$$

Thus by Eq. (4.9) such terms do not arise in Eq. (4.8). A similar argument can be used to eliminate the terms  $\chi_{:a_1 a_1 a_2 a_2 \dots} S^k$  from Eq. (4.8).

Extend  $S$  to be covariant constant along the geodesic normal rays from the boundary. This permits us to move covariant derivatives outside the trace once again. We apply Lemma 3.7 to see

$$\sum_p (-1)^p \text{Tr}_{\Lambda^p(M)} \{ES^k\} = 0 \quad \text{for } k < m - 3.$$

Thus exactly one covariant derivative of  $E$  can appear and Eq. (4.8) becomes

$$P_{m+1}(g)(0) = \bar{c}_{m+1} \sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^m)} \{E;_m S^{m-3}\}(0).$$

If  $m = 3$ , then  $\sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^3)} \{E\} = 0$ . This implies

$$\sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^3)} \{E;_m\} = 0$$

and hence  $c_{m+1} = 0$  as desired.

Suppose that  $m \geq 4$ . Since  $S$  vanishes on  $\Lambda(\mathbb{R}^{m-1})^\perp$ , we have

$$P_{m+1}(g)(0) = \bar{c}_{m+1} \sum_p (-1)^p \text{Tr}_{\Lambda^p(\mathbb{R}^{m-1})} \{E;_m S^{m-3}\}.$$

We may decompose  $\Lambda(\mathbb{R}^{m-1}) = \Lambda(\mathbb{R}^2) \otimes \Lambda(\mathbb{R}^{m-3})$  to express  $E;_m = \tilde{E} \otimes \text{Id}$  and  $S = \text{Id} \otimes \tilde{S}$ . This then leads to the corresponding decomposition of the supertrace:

$$\begin{aligned} &\sum_p (-1)^p \text{Tr}_{\Lambda^p \mathbb{R}^{m-1}} \{E;_m S^{m-3}\} \\ &= \sum_a (-1)^a \text{Tr}_{\Lambda^a(\mathbb{R}^2)} \{\tilde{E};_m\} \cdot \sum_b (-1)^b \text{Tr}_{\Lambda^b(\mathbb{R}^{m-3})} \{\tilde{S}^{m-3}\}. \end{aligned}$$

The computation performed above shows that the supertrace of  $S^{m-3}$  on  $\mathbb{R}^{m-3}$  is  $(-1)^{m-3}(m-3)!A_1 \cdots A_{m-3}$ . A direct calculation of the supertrace of  $E;_m$  on  $\mathbb{R}^2$  yields  $2A_0$ . The final assertion of Lemma 4.2 now follows.  $\square$

We continue our study by using the various functorial properties to show the following lemma.

**Lemma 4.3.**

- (1)  $\bar{c}_{n,m,k}^i = (4\pi)^{-(m-1)/2} \bar{c}_{n,1,k}^i$ .
- (2) If  $n \geq 3$ , then  $\bar{c}_{n,m,1}^0 = (1/2)\bar{c}_{n,m,0}^0$ .
- (3) If  $n \geq 5$ , then  $\bar{c}_{n,m,0}^3 = \bar{c}_{n-2,m,1}^0$ .

To prove Assertion (1), we use product formulas. Let  $M_1 = T^{m-1}$  be the torus and let  $D_1$  be the scalar Laplacian. Since the structures are flat,

$$a_{n,m-1}(x_1, D_1) = \begin{cases} (4\pi)^{-(m-1)/2} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Let  $(M_2, D_2) = ([0, 1], -\partial_r^2)$ . Let  $M = M_1 \times M_2$  and  $D = D_1 + D_2$ . Let  $\mathcal{B} = \nabla_{e_m} + S$  where  $S$  is constant and where  $e_m$  is the inward unit normal;  $e_m = \partial_r$  when  $r = 0$  and  $e_m = -\partial_r$  when  $r = 1$ . An analogous argument to that which was used to establish Lemma 2.3 (4)

can be used to establish the following identity from which Assertion (1) follows:

$$\begin{aligned} a_{n,m,k}(y, D, \mathcal{B}) &= \sum_{n_1+n_2=n} a_{n_1,m-1}(x_1, D_1) \cdot a_{n_2,1,k}(y_2, D_2, \mathcal{B}) \\ &= (4\pi)^{-(m-1)/2} a_{n,1,k}(y_2, D_2, \mathcal{B}). \end{aligned}$$

In view of Assertion (1), it suffices to take  $m = 1$  in the proof of the remaining assertions. We use results from [6]. Let  $M := [0, 1]$  and let  $D_0 := -\partial_r^2$ . We choose  $f$  so that  $f$  vanishes identically near  $r = 1$  so only the component  $r = 0$  where  $\partial_r$  is the inward unit normal is relevant. To prove Assertion (2), we consider a conformal variation  $D_\varepsilon := e^{-2\varepsilon f} D_0$ . Then

$$\partial_\varepsilon S|_{\varepsilon=0} = -\frac{1}{2} f_{;m} \quad \text{and} \quad \partial_\varepsilon a_n(1, D_\varepsilon)|_{\varepsilon=0} = (1 - n)a_n(f, D_0).$$

For  $n \geq 3$ ,  $f_{;m} S^{n-2}$  arises from no other term. Thus we may show  $\bar{c}_{n,1,0}^0 = (1/2)\bar{c}_{n,1,0}^0$  by computing:

$$\begin{aligned} \partial_\varepsilon a_n(1, D_\varepsilon) &= \partial_\varepsilon \int_{\partial M} \bar{c}_{n,1,0}^0 S^{n-1} dy|_{\varepsilon=0} + \dots \\ &= -\frac{1}{2}(n - 1)\bar{c}_{n,1,0} \int_{\partial M} f_{;m} S^{n-2} dy + \dots \\ &= (1 - n)a_n(f, D_0) = (1 - n) \int_{\partial M} f_{;m} S^{n-2} dy + \dots . \end{aligned}$$

To prove Assertion (3), we consider a scalar variation  $D_\varrho := D_0 - \varrho f$ . We have

$$\partial_\varrho a_n(1, D_\varrho)|_{\varrho=0} = a_{n-2}(f, D_0).$$

If  $n \geq 5$ , then this is the only way a term involving  $f_{;m} S^{n-4}$  can arise. We show  $\bar{c}_{n,1,0}^3 = \bar{c}_{n-2,1,1}^0$  by computing:

$$\begin{aligned} \partial_\varrho a_n(1, D_\varrho)|_{\varrho=0} &= \partial_\varepsilon \int_{\partial M} \bar{c}_{n,1,0}^3 E_{;m} S^{n-4} dy|_{\varrho=0} + \dots = a_{n-2}(f, D_0) \\ &= \int_{\partial M} \bar{c}_{n-2,1,1}^0 f_{;m} S^{n-4} dy + \dots . \end{aligned}$$

**Remark.** Lemma 4.3 (2) fails if  $n = 2$  and Lemma 4.3 (3) fails if  $n = 4$  as there are interior terms which also contribute to the variational formulae.

### 5. Proof of Theorem 1.2

We use Lemmas 4.1–4.3 to determine the constants of Lemma 1.3:

$$\begin{aligned}
 c_{m+1,m} &= \frac{1}{\sqrt{\pi}8^{\bar{m}}\pi^{\bar{m}}\bar{m}!} \quad \text{for } m = 2\bar{m} + 1, \\
 c_{m+1,m,1}^k &= \frac{1}{\pi^k 8^k k!} c_{m-2k+1,m-2k,1}^0 = \frac{1}{\pi^k 8^k k!} \bar{c}_{m-2k+1,m-2k,1}^0 \\
 &= \frac{1}{2\pi^k 8^k k!} \bar{c}_{m-2k+1,m-2k,0}^0 = \frac{2\sqrt{\pi}}{2\pi^k 8^k k!} \bar{c}_{m-2k+1,m-2k+1,0}^0 \\
 &= \frac{\sqrt{\pi}}{8^k \pi^k k! \operatorname{vol}(S^{m-2k})(m-2k)!}, \\
 c_{m+1,m,0}^{1,k} &= \frac{1}{\pi^k 8^k k!} c_{m+1-2k,m-2k,0}^{1,0} = \frac{1}{\sqrt{\pi} \pi^k 8^k k! \operatorname{vol}(S^{m-2k-2})(m-2k-2)!}, \\
 c_{m+1,m,0}^{2,k} &= \frac{1}{\pi^k 8^k k!} c_{m+1-2k,m-2k,0}^{2,0} = 0, \\
 c_{m+1,m,0}^{3,k} &= \frac{1}{\pi^k 8^k k!} c_{4,3,0}^{3,0} = 0 \quad \text{for } 2k = m - 3, \\
 c_{m+1,m,0}^{3,k} &= \frac{1}{\pi^k 8^k k!} c_{m-2k+1,m-2k,0}^{3,0} = \frac{1}{\pi^k 8^k k!} \bar{c}_{m-2k+1,m-2k,0}^3 \\
 &= \frac{1}{\pi^k 8^k k!} \bar{c}_{m-2k-1,m-2k,0}^0 = \frac{1}{2\sqrt{\pi} \pi^k 8^k k!} \bar{c}_{m-2k-1,m-2k-1,0}^0 \\
 &= \frac{1}{2\sqrt{\pi} \pi^k 8^k k!} c_{m-2k-1,m-2k-1,0}^0 \\
 &= \frac{1}{2\sqrt{\pi} \pi^k 8^k k! \operatorname{vol}(S^{m-2k-2})(m-2k-2)!} \quad \text{for } 2k < m - 3.
 \end{aligned}$$

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